Math 241

Problem Set 8 solution manual

Exercise. A8.1

a- If v, and v' are in the same orbit then there exist $A \in SO(3)$ such that $v = Av'$, and hence since $A \in SO(3)$ then A preserves the norm and hence v, and v' have the same norm.

Now suppose v, and v' have the same norm say $||v|| = ||v'|| = c$. Then

Consider $u = \frac{v}{||v||}$(i.e $cu = v$), and extend it into an orthonormal basis of $\mathbb{R}^3$ $\{u, u_2, u_3\}$. (see part (b) for a method for extending u into orthonormal basis. Also let $u' = \frac{v'}{||v'||}$ (i.e $u'c = v'$), and extend it into an orthonormal basis $\{u', u_2', u_3'\}$.
Let $A_1 \in M_{n\times n}(\mathbb{R})$ be such that :
- $A_1(u) = u'$
- $A_1(u_2) = u_2'$
- $A_1(u_3) = u_3'$

Notice that A_1 transfers v into v', since $A_1(cu) = cA(u) = cu' = v'$. Moreover, since A_1 is a transition matrix between two orthonormal basis then $A_1 \in O(3)$. If $|A_1| = 1$ we are done, else we consider the following transformation:
- $A_2(u) = u'$
- $A_2(u_2) = u_2'$
- $A_2(u_3) = -u_3'$

Then similarly A_2 transfers v into v', and $A_2 \in O(3)$, and in case $|A_1| = -1$, then $|A_2| = +1$

b- It is easy to see directly that $A' = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ works.

As for A'' we have to consider the two vector $(2, 4, 1)$ and $(4, 2, 1)$, and we have to find an orthonormal basis for $\mathbb{R}^3$ $\{u_1, u_2, u_3\}$ such that u_1 is in the direction of $(2, 4, 1)$,and u_1, u_2 span the same space spanned by v , and v'.
To do so we proceed as follows :
Let $u_1 = \frac{1}{\sqrt{21}}(2, 4, 1)$, where $\sqrt{21} = |v|$.
Next to find u_2 we use Gram Schmidt method, i.e we let $v_2 = v' - \frac{<v,v'>}{||v||||v'||}v = (4, 2, 1) - \frac{17}{21}(2, 4, 1)$, then $21.v_2 = (50, -26, 4)$. Hence we reduce it to the vector (25,-13,2), whose unit vector is $u_2 = \frac{1}{\sqrt{798}}(25, -13, 2)$.
Finally we have to choose $u_3 = (a, b, c)$ to be a unit vector orthogonal to u_1, u_2, or equivalently orthogonal to v, v'. So we consider the following system:

$$\begin{cases} 4a + 2b + c = 0 \\ 2a + 4b + c = 0 \end{cases} \implies \begin{cases} 2b + c = -4a \\ 4b + c = 0 \end{cases} \implies b = a \text{ and } c = -6a.$$

So the unit vector u_3 would be equal to $\frac{1}{38}(1, 1, -6)$.

Hence we have our orthonormal basis $\{u_1, u_2, u_3\}$

Now consider the figure 1 : and notice that $v' = cos(\theta)u_1 + \sin(\theta)u_2$ (this angle θ can be computed using the formula $cos(\theta) = \frac{<v,v'>}{||v||||v'||} = \frac{17}{21}$), and so we can consider the matrix $B' = \begin{bmatrix} cos(\theta) & -sin(\theta) & 0 \\ sin(\theta) & cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{17}{21} & -\frac{2}{21}\sqrt{38} & 0 \\ \frac{2}{21}\sqrt{38} & \frac{17}{21} & 0 \\ 0 & 0 & 1 \end{bmatrix}$, this matrix transfer the vector $(c, 0, 0)$ into the vector $(cos(\theta), sin(\theta), 0)$, i.e it transfers v into v' if they were expressed according to the basis $\{u_1, u_2, u_3\}$. To go back to our initial basis we have to find the transition matrix P that transfers the basis $\{u_1, u_2, u_3\}$ to the canonical basis. Then our $A' = PB'P^{-1}$, to find P we just have to write the vector u_i in terms of the canonical basis; i.e $P = \begin{bmatrix} \frac{2}{\sqrt{21}} & \frac{25}{\sqrt{798}} & \frac{1}{\sqrt{38}} \\ \frac{4}{\sqrt{21}} & -\frac{3}{\sqrt{798}} & \frac{1}{\sqrt{38}} \\ \frac{1}{\sqrt{21}} & \frac{2}{\sqrt{798}} & -\frac{6}{\sqrt{38}} \end{bmatrix}$, and hence $P^{-1} = \begin{bmatrix} \frac{4}{167}\sqrt{21} & \frac{38}{167}\sqrt{21} & \frac{7}{167}\sqrt{21} \\ \frac{25}{668}\sqrt{798} & -\frac{13}{668}\sqrt{798} & \frac{1}{334}\sqrt{798} \\ \frac{11}{668}\sqrt{38} & \frac{21}{668}\sqrt{38} & -\frac{53}{334}\sqrt{38} \end{bmatrix}$

Hence Our $A = P.B.P^{-1}$.

c- Let $u_1 = \frac{v}{|v|}$, and extend u_1 to an orthonoraml basis $\{u_1, u_2, u_3\}$ for $\mathbb{R}^3$. Then consider the correspondence between $\mathbb{R}/(2\pi\mathbb{Z})$ and the matrices that stabilize v, for any $\theta \in \mathbb{R}/(2\pi\mathbb{Z})$ we consider the matrix: $A_\theta = \begin{bmatrix} 1 & 0 & 0 \\ 0 & cos(\theta) & -sin(\theta) \\ 0 & sin(\theta) & cos(\theta) \end{bmatrix}$. Notice that for any $\theta \in \mathbb{R}/(2\pi\mathbb{Z})$, A_θ stabilizes v (if v is written in the coordinates corresponding to the basis $\{u_1, u_2, u_3\}$), and any matrix $M \in SO(3)$ that stabilize v must be a rotation about v with an angle θ since M only transfers orthonormal basis into orthonormal basis , and hence if it takes u_1 into u_1 it can only rotate the other two vectors by some angle.

Exercise. A8.2

a- Let M and M' be in the same orbit, then $\exists\ g \in GL(n, \mathbb{R})$ such that $M' = gMg^{-1}$, then the characteristic polynomial of M' is the determinant of $M' - \lambda I$, it is equal to $|M' - \lambda(gg^{-1})| = |gMg^{-1} - g(\lambda I)g^{-1}| = |g(M - \lambda I)g^{-1}| = |g||M - \lambda I||g^{-1}| = |M - \lambda I|$, and hence M, and M' have the same characteristic polynomial.

b- Suppose M' is in the orbit of M, then by part (a) we know that M' has the same characteristic polynomial as M. On the other hand, suppose M' has the characteristic polynomial $(x - 2)(x - 3)$ then M' is diagonalizable with eigen values 2, and 3. Hence $\exists\ g \in GLn(\mathbb{R})$ such that $M' = gMg^{-1}$, so M' belongs to the orbit of M.

Now to find the stabilizer of M consider the matrix $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

If $g \in Stab(M)$ then $gMg^{-1} = M \implies gM = Mg$, and hence :

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

$$\implies \begin{bmatrix} 2a & 2b \\ 3c & 3d \end{bmatrix} = \begin{bmatrix} 2a & 3b \\ 2c & 3d \end{bmatrix} \implies 2b = 3b, \text{ and } 3c = 2c \implies b = c = 0$$

$\implies$ For g to be a stabilizer of M, g must be a diagonal matrix.

c- It is easy to see using part (a) that the orbit of M is contained in the set of matrices of characteristic polynomial $(x-1)^2$. Now consider the matrix $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Notice that for any $g \in Gl(n, \mathbb{R})$ $gIg^{-1} = I$, and hence the orbit of I is the singleton I, so M can't be in the orbit of I, which is equivalent to saying I is not in the orbit of M.

So we deduce that we can find a Matrix (I) with characteristic polynomial $(x-1)^2$ which is not in the orbit of M. Hence the orbit of M is a proper subset of the matrices of characteristic polynomial $(x-1)^2$.

Now to find the stabilizer of M , again we let $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

If $g \in Stab(M)$ then $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$\implies \begin{bmatrix} a & a+b \\ c & c+d \end{bmatrix} = \begin{bmatrix} a+c & b+d \\ c & d \end{bmatrix}$.

So we have $c = 0$, $a = d$.

So the $Stab(M) = \{ \begin{bmatrix} a & b \\ 0 & a \end{bmatrix} \mid a, b \in \mathbb{R}\}$

Finally Let $T \neq I$ be such that T has the characteristic polynomial $(x-1)^2$, then there exists a vector $v \neq 0$ such that $Tv = v$. Extend v into a basis for $\mathbb{R}^2$ $\{v, w\}$. Then $Tw = aw + bv$ and hence $[T]_{\{v,w\}} = \begin{bmatrix} 1 & b \\ 0 & a \end{bmatrix}$. But since we know that the characteristic polynomial of T is $(x-1)^2$, then a must be 1. On the other hand choosing the basis $\{bv, w\}$ will show that our initial T is conjugate to $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

So we deduce that we only have to orbits with characteristic polynomial $(x-1)^2$.

d- Let M be a matrix of characteristic polynomial $x^2 + 1$, then we have by Cayley-Hamilton, $M^2 + I = 0$, and hence $M(Mv) = -v$ for all $v \neq 0$. Then Notice that if $w = Mv$ is linearly independent to v, or else if $w = \lambda v$ we get that λ is an eigen value for M, and hence a solution for $x^2 + 1$, which is impossible. Then we get the following system :
$Mv = w$
$Mv = -v$
And hence M is conjugate to the matrix $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

Hence all Matrices of characteristic polynomial $x^2 + 1$ are conjugate to $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

e- We can only find one orbit with characteristic polynomial $(x-\lambda_1)(x-\lambda_2)...(x-\lambda_n)$, since any matrix having this characteristic polynomial is diagonalizable since all the eigen vectors are distinct, and hence it should be conjugate to the matrix $\begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ : & 0 & \ddots & 0 \\ 0 & \dots & 0 & \lambda_n \end{bmatrix}$.

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Lemma 1. Rank(PAQ)=Rank(A) for P, and Q invertible.

proof. Let us show that Rank(AQ)=Rank(A). By definition we know that the rank of AQ is the dimension of its image, and hence since Q is invertible we can easily see that the image of AQ is of the same dimension as the image of A. On the other hand, by rank nullity theorem, since $Ker(PA) = Ker(A)$ we get that Rank(PA)=Rank(A), combine the two results to get the lemma. (Note that you when you apply Rank(PA)=Rank(A), your A represents AQ)

Now back to the problem:

By above lemma we know that any two conjugate matrices have the same rank, so we only need to see why the third and the fourth matrix are not conjugate. Notice that if two matrices are conjugate then their squares are too, and hence if since the square of the third matrix is zero while the square of the fourth matrix is of rank one we get our result.

Section. 16

Exercise. 2
$Stab(1) = G_1 = \{g \in G \mid g.1 = 1\} = \{\rho_0, \delta_2\}$.
$Stab(2) = G_2 = \{g \in G \mid g.2 = 2\} = \{\rho_0, \delta_1\}$.
$Stab(3) = Stab(1)$
$Stab(4) = Stab(2)$
$Stab(s_1) = Stab(s_3) = \{\rho_0, \mu_1\}$.
$Stab(s_2) = Stab(s_4) = \{\rho_0, \mu_2\}$.
$Stab(m_1) = Stab(m_2) = \{\rho_0, \rho_2, \mu_1, \mu_2\}$.
$Stab(d_1) = Stab(d_2) = \{\rho_0, \rho_2, \delta_1, \delta_2\}$.
$Stab(p_1) = Stab(p_3) = \{\rho_0, \mu_1\}$.
$Stab(p_2) = Stab(p_4) = \{\rho_0, \mu_2\}$.

Exercise. 3

The orbits of this action are :
$\{1, 2, 3, 4\}$
$\{s_1, s_2, s_3, s_4\}$
$\{m_1, m_2\}$
$\{d_1, d_2\}$
$\{p_1, p_2, p_3, p_4\}$
$\{c\}$

Exercise. 5

A group G is said to be transitive if it has only one orbit, i,e for all $x, y \in G$, there exists a $g \in G$ such that $x = g.y$.

Exercise. 6

A sub-G-set is a set that contains all the orbits of its elements, i.e A is a sun-G-set if for all $x \in A$, $\mathcal{O}(x) \in A$. In this case A must be a union of some orbits of G.

Exercise. 9

a- Let $U = \{s_1, s_2, s_3, s_4\}$, and $V = \{p_1, p_2, p_3, p_4\}$.

Define $f : U \longrightarrow V$ to be such that $f(s_i) = p_i$.

It is clear that f is a bijection. And since the elements of D_4 acts on $\{s_1, s_2, s_3, s_4\}$ the same way it acts on $\{p_1, p_2, p_3, p_4\}$ f is an isomorphism

b- Suppose we have an isomorphism g between $\{1, 2, 3, 4\}$ and $\{s_1, s_2, s_3, s_4\}$, then $g(1) = s_i$ for some $0 \leq i \leq 4$, but $\delta_2.g(1) \neq s_i$ and $g(\delta_2.1) = g(1) = s_i$. Hence $\{1, 2, 3, 4\}$ is not isomorphic to $\{s_1, s_2, s_3, s_4\}$.

c- By the same argument of part (b) we can see that the only two G_sets that are isomorphic are $\{s_1, s_2, s_3, s_4\}$, $\{p_1, p_2, p_3, p_4\}$.

Exercise. 13

a- It is easy to see that when $g \in \mathbb{R}$ acts on an element $x \in \mathbb{R}^2$ it gives another image in $\mathbb{R}^2$ by the definition of rotation. (if $x = r(cos(\theta), sin(\theta))$ the $g.x = r(cos(\theta + g), sin(\theta + g)) \in \mathbb{R}^2$).

b- The orbit of p is the circle passing throught p of center (0,0).

c- $Stab(g) = G_p = \{g \in \mathbb{R} \mid g.p = p\} = \{g \in \mathbb{R} \mid (cos(\theta + g) = cos(\theta) \text{ and } sin(\theta + g)sin(\theta)\} = \{g = 2k\pi \mid k \in \mathbb{Z}\}$.